Ex. : Solve $\frac{d^4 y}{dx^4} + k^4 y = 0.$ (M.

The auxiliary equation is $D^4 + k^4 = 0$.

$$\therefore \quad (D^4 + 2D^2k^2 + k^4) - (2D^2k^2) = 0$$

$$\therefore \quad (D^2 + k^2)^2 - (\sqrt{2} \cdot Dk)^2 = 0$$

$$\therefore \quad (D^2 - \sqrt{2} \cdot Dk + k^2)(D^2 + \sqrt{2} \cdot Dk + k^2) = 0$$

Now, $D^2 - \sqrt{2} \cdot Dk + k^2 = 0$ gives $D = \frac{k \pm ik}{\sqrt{2}}$

$D^2 + \sqrt{2} \cdot Dk + k^2 = 0$ gives $D = \frac{-k \pm ik}{\sqrt{2}}$

Since, we have two pairs of complex roots, the solution is

$$y = e^{(k/\sqrt{2})x}\left[c_1 \cos(k/\sqrt{2})x + c_2 \sin(k/\sqrt{2})x\right] + e^{(-k/\sqrt{2})x}\left[c_3 \cos(k/\sqrt{2})x + c_4 \sin(k/\sqrt{2})x\right]$$

Ex. 2 : Solve : $(D^2 - D - 2)\,y = 2\log x + \frac{1}{x} + \frac{1}{x^2}$ **(June 2000)**

Sol. :

Step I : A. E. is $D^2 - D - 2 = 0$

Roots : 2 , – 1

$\therefore$ C. F. is $y = c_1 e^{2x} + c_2 e^{-x}$. ... (1)

Step II : P. I. is given by,

$$y = \frac{1}{(D-2)(D+1)}\left[2\log x + \frac{1}{x} + \frac{1}{x^2}\right] \text{ by Partial fractions}$$

$$= \left[\frac{1}{3}\frac{1}{(D-2)} - \frac{1}{3(D+1)}\right]\left(2\log x + \frac{1}{x} + \frac{1}{x^2}\right)$$

$$= \frac{1}{3}\frac{1}{(D-2)}\left(2\log x + \frac{1}{x} + \frac{1}{x^2}\right) - \frac{1}{3}\frac{1}{(D+1)}\left(2\log x + \frac{1}{x} + \frac{1}{x^2}\right)$$

$$\left(\text{using } \frac{1}{D-M}X = e^{mx}\int e^{-mx}\,.\,X.\,dx\right)$$

$$= \frac{1}{3}e^{2x}\int e^{-2x}\left(2\log x + \frac{1}{x} + \frac{1}{x^2}\right).dx - \frac{1}{3}e^{-x}\int e^{x}\left(2\log x + \frac{1}{x} + \frac{1}{x^2}\right)dx$$

Putting $-2x = t$ in the first integral, we get

$$= \frac{1}{3}e^{2x}\int e^{t}\left[2\log\left(-\frac{t}{2}\right) - \frac{2}{t} + \frac{4}{t^2}\right]\left(-\frac{dt}{2}\right) - \frac{1}{3}e^{-x}\int e^{x}\left(2\log x + \frac{1}{x} + \frac{1}{x^2}\right)dx$$

$$= -\frac{1}{3}e^{2x}\int e^{t}\left[\left\{\log\left(\frac{-t}{2}\right) - \frac{2}{t}\right\} + \left\{\frac{1}{t} + \frac{2}{t^2}\right\}\right] dt^{x}\int e^{x}\left[\left(2\log x - \frac{1}{x}\right) + \left(\frac{2}{x} + \frac{1}{x^2}\right)\right]dx$$

We make the arrangements of the terms so that we can use

$$\int e^{x}\left[f(x) + f'(x)\right]dx = e^{x} f(x) = -\frac{1}{3}e^{2x}\int\left[e^{t}\left\{\log\left(\frac{-t}{2}\right)\right\} - \frac{2}{t}\right] - \frac{1}{3}e^{-x}\left[e^{x}\left(2\log x - \frac{1}{x}\right)\right.$$

$$= -\frac{1}{3}e^{2x}\left[e^{-2x}\left\{\log x + \frac{1}{x}\right\}\right] - \frac{1}{3}e^{-x}\left[e^{x}\left\{2\log x - \frac{1}{x}\right\}\right]$$

$\because -2x = t$

$$= -\frac{1}{3}\left\{\log x + \frac{1}{x}\right\} - \frac{1}{3}\left\{2\log x - \frac{1}{x}\right\} = -\log x \quad \text{... (2)}$$

From (1) and (2), G.S. is $y = c_1 e^{-x} + c_2 e^{2x} - \log x$.

Ex. 3 : Solve : $(D^2 + 5D + 6)\, y = e^{-2x} \sec^2 x\,(1 + 2\tan x)$ (I

Sol. : Clearly , C. F. is

$$y = c_1 e^{-2x} + c_2 e^{-3x}$$

P. I. is

$$y = \frac{1}{(D+2)(D+3)} e^{-2x} \cdot \sec^2 x\,(1 + 2\tan x)$$

$$= \left[\frac{1}{D+2} - \frac{1}{D+3}\right]\left[e^{-2x} \sec^2 x(1 + 2\tan x)\right]$$

$$= e^{-2x} \int e^{2x} e^{-2x} \sec^2 x\,(1 + 2\tan x) \cdot dx$$

$$- e^{-3x} \int e^{3x} e^{-2x} \sec^2 x\,(1 + 2\tan x) \cdot dx$$

$$= e^{-2x} \frac{1}{2} \int \underset{f'}{(2\sec^2 x)}\ \underset{f}{(1 + 2\tan x)} \cdot dx$$

$$- e^{-3x} \int e^x [\underset{f}{\sec^2 x} + \underset{f'}{2\tan x \cdot \sec^2 x}] \cdot dx$$

[using $\int f^1 f' = \frac{f^2}{2}$ and $\int e^x (f + f') = e^x . f(x)$]

$$= \frac{1}{2} e^{-2x}\left[\frac{(1 + 2\tan x)^2}{2}\right] - e^{-3x} \cdot e^x \cdot \sec^2 x$$

$$= \frac{1}{4} e^{-2x} [1 + 4\tan x + 4\tan^2 x - 4\sec^2 x]$$

$$= \frac{1}{4} e^{-2x} [4\tan x - 3]$$

From (1) and (2),

G. S. is $y = c_1 e^{-2x} + c_2 e^{-3x} + \frac{1}{4} e^{-2x}(4\tan x - 3)$

$(D^2 + 3D + 2)\, y = \sin e^x$. **(June 2000)**

I: We find C.F. Auxiliary Equation is $D^2 + 3D + 2 = 0$

Roots are $-2, -1$ (real and distinct)

$\therefore$ C.F. is $y = c_1 e^{-2x} + c_2 e^{-x}$ ... (i)

II:

P. I. is $y = \dfrac{1}{(D+1)(D+2)} \sin e^x$ by Partial fraction,

$$= \left[\frac{1}{D+1} - \frac{1}{D+2}\right]\sin(e^x) = \frac{1}{D+1}\sin(e^x) - \frac{1}{(D+2)}\sin(e^x)$$

$$= e^{-x}\int e^x \sin(e^x)\,.\,dx - e^{-2x}\int e^{2x}.\sin(e^x)\,.\,dx$$

Using $\dfrac{1}{(D+m)} X = e^{-mx}\int e^{mx}.X.\,dx$

Let $e^x = t \quad \therefore e^x\,dx = dt$

$$\therefore \quad y = e^{-x}\int \sin t\,dt - e^{-2x}\int t.\sin t.\,dt$$

$$= e^{-nx}(-\cos t) - e^{-2nx}[t(-\cos t) - (1)(-\sin t)]$$

$$= -e^{-x}\cos(e^x) - e^{-2n}\left[-e^x\cos(e^x) + \sin(e^x)\right]$$

$$= -e^{-2x}\sin e^x$$

$\therefore$ G. S. is $y = c_1 e^{-2x} + c_2 e^{-x} - e^{-2x}\sin(e^x)$

~~Ex. 4~~ 5: Solve $(D^2 + a^2)\, y = \sec ax$.

: The auxiliary equation is $D^2 + a^2 = 0 \quad \therefore \quad D = +ai, -ai$.

$\therefore$ The C.F. is $y = c_1 \cos ax + c_2 \sin ax$.

$$\text{P.I.} = \frac{1}{D^2 + a^2} \sec ax = \frac{1}{(D + ai)(D - ai)} \sec ax$$

$$= \frac{1}{2ai} \left[\frac{1}{D - ai} - \frac{1}{D + ai} \right] \sec ax$$

$$= \frac{1}{2ai} \left[\frac{1}{D - ai} \sec ax - \frac{1}{D + ai} \sec ax \right]$$

$$= \frac{1}{2ai}\left[e^{aix}\int e^{-aix}\sec ax\,dx - e^{-aix}\int e^{aix}\sec ax\,dx\right]$$

$$= \frac{1}{2ai}\left[e^{aix}\int(\cos ax - i\sin ax)\sec ax\,dx\right.$$

$$\left. - e^{-aix}\int(\cos ax + i\sin ax)\sec ax\,dx\right]$$

$$= \frac{1}{2ai}\left[e^{aix}\int(1 - i\tan ax)\,dx - e^{-aix}\int(1 + i\tan ax)\,dx\right]$$

$$= \frac{1}{2ai}\left[e^{aix}\left\{x - \frac{i}{a}\log\sec ax\right\} - e^{-aix}\left\{x + \frac{i}{a}\log\sec ax\right\}\right]$$

$$= \frac{1}{2ai}\left[(\cos ax + i\sin ax)\left\{x - \frac{i}{a}\log\sec ax\right\}\right.$$

$$\left. - (\cos ax - i\sin ax)\left\{x + \frac{i}{a}\log\sec ax\right\}\right]$$

$$= \frac{1}{2ai}\left\{2\,i\,x\sin ax - \frac{2i}{a}\cos x\log\sec ax\right\}$$

$$= \frac{x}{a}\sin ax - \frac{1}{a^2}\cos ax\log\sec ax$$

$$= \frac{x}{a}\sin ax + \frac{1}{a^2}\cos ax\log\cos ax$$

∴ The complete solution is

$$y = c_1\cos ax + c_2\sin ax + \frac{x}{a}\sin ax + \frac{1}{a^2}\cos ax\log\cos ax.$$

Ex. 2 : Solve $(D^2 + a^2) = 2a \tan ax$.

Sol. : The auxiliary equation is $D^2 + a^2 = 0 \quad \therefore \quad D = ai, -ai$.

$\therefore$ The C.F. is $y = c_1 \cos ax + c_2 \sin ax$.

$$\text{P.I.} = \frac{2a}{D^2 + a^2} \tan ax = \frac{1}{i}\left[\frac{1}{D - ai} - \frac{1}{D + ai}\right] \tan ax$$

Now, $$\frac{1}{D - ai} \tan ax = e^{aix} \int e^{-aix} \tan ax \, dx$$

$$= e^{aix} \int (\cos ax - i \sin ax) \tan ax \, dx$$

$$= e^{aix} \int \left(\sin ax - i \frac{\sin^2 ax}{\cos ax} \right) dx$$

$$= e^{aix} \int \left(\sin ax - i \frac{(1 - \cos^2 ax)}{\cos ax} \right) dx$$

$$= e^{aix}\int(\sin ax - i\sec ax + i\cos ax)\,dx$$

$$= e^{aix}\left[-\frac{\cos ax}{a} - \frac{i}{a}\log\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right) + \frac{i}{a}\sin ax\right]$$

$$= -e^{aix}\left[\frac{1}{a}(\cos ax - i\sin ax) + \frac{i}{a}\log\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)\right]$$

$$= -\frac{e^{aix}}{a}\left[e^{-aix} + \frac{i}{a}\log\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)\right]$$

$$= -\frac{1}{a}\left[1 + i\,e^{aix}\log\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)\right]$$

Changing i to $-i$,

$$\frac{1}{D+ai}\tan ax = -\frac{1}{a}\left[1 - i\,e^{-aix}\log\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)\right]$$

$$\text{P.I.} = \frac{1}{i}\left[-\frac{i}{a}\left(e^{aix}+e^{-aix}\right)\log\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)\right]$$

$$= -\frac{2}{a}\left(\frac{e^{aix}+e^{-aix}}{2}\right)\log\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)$$

$$= -\frac{2}{a}\cos ax\log\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)$$

∴ The complete solution is

$$y = c_1\cos ax + c_2\sin ax - \frac{2}{a}\cos ax\log\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right).$$

Ex. : Solve $(D^2 - 6D + 9)\,y = e^{3x}(1 + x)$.

: The auxiliary equation is $D^2 - 6D + 9 = 0$

$\therefore \quad (D-3)^2 = 0 \qquad \therefore \quad D = 3, 3.$

$\therefore$ The C.F. is $y = (c_1 + c_2 x)\,e^{3x}$.

$$\text{P.I.} = \frac{1}{D^2 - 6D + 9} e^{3x}(1 + x)$$

$$= \frac{1}{(D-3)^2} e^{3x} + \frac{1}{(D-3)^2} e^{3x} \cdot x$$

$$= \frac{x^2}{2!} e^{3x} + e^{3x} \cdot \frac{1}{(D+3-3)^2} x$$

$$= \frac{x^2}{2} e^{3x} + e^{3x} \cdot \frac{1}{D^2} x$$

But $\frac{1}{D^2} x = \frac{1}{D} \int x\, dx = \frac{1}{D} \frac{x^2}{2}$

$$= \int \frac{x^2}{2}\, dx = \frac{x^3}{6}$$

$\therefore$ The complete solution is

$$y = (c_1 + c_2 x) e^{3x} + \frac{x^2}{2} e^{3x} + \frac{x^3}{6} \cdot e^{3x}.$$

Ex. 13 : $(D^2 - 4D + 4)\,y = 8x^2 \cdot e^{2x} \cdot \sin 2x$ (Dec. 97, 6 M

Sol. : A.E. is $D^2 - 4D + 4 = 0$ Roots : 2, 2

$\therefore$ C.F. is $y = (c_1 x + c_2)\, e^{2x}$

P.I. is $y = \frac{1}{(D-2)^2}\, 8x^2 \cdot e^{2x} \cdot \sin 2x$

$$= 8e^{2x} \frac{1}{(D+2-2)^2} x^2 \sin 2x \quad \left(\text{Using } \frac{1}{f(D)} e^{ax} \cdot V = e^{ax} \frac{1}{f(D+a)} V\right)$$

$$= 8e^{2x} \frac{1}{D^2} x^2 \sin 2x = 8e^{2x} \frac{1}{D}\left[\frac{1}{D} x^2 \sin 2x\right] = 8e^{2x} \frac{1}{D}\left[\int x^2 \sin 2x \cdot dx\right]$$

$\because \frac{1}{D} X = \int x \cdot dx$ using the generalised formula of integration by parts, we get

$$= 8e^{2x} \frac{1}{D}\left[x^2\left(-\frac{\cos 2x}{2}\right) - (2x)\left(-\frac{\sin 2x}{4}\right) + (2)\left(\frac{\cos 2x}{8}\right)\right]$$

Again applying the formula,

$$= 8e^{2x}\left[-\frac{1}{2}\left\{x^2\left(\frac{\sin 2x}{2}\right)(-2x)\left(\frac{-\cos 2x}{4}\right) + (2)\left(-\frac{\sin 2x}{8}\right)\right\} + \frac{1}{2}\left\{x\left(-\frac{\cos 2x}{2}\right) - (1)\left(-\frac{\sin 2x}{4}\right)\right\} + \frac{1}{4}\left(\frac{\sin 2x}{2}\right)\right]$$

$$= 8e^{2x}\left[-\frac{x^2}{4}\sin 2x - \frac{x}{2}\cos 2x + \frac{3}{8}\sin 2x\right]$$

$\therefore$ G.S. is $y = C.F + P.I.$

Ex. 8 : $\dfrac{d^3y}{dx^3} - y = (1 + e^x)^2$. (Dec. 95

Sol. :

$$\text{Let } D \equiv \frac{d}{dx};$$

$$\therefore (D^3 - 1)y = (1 + e^x)^2$$

$$\text{A.E. is } D^3 - 1 = 0$$

$$\therefore (D - 1)(D^2 + D + 1) = 0$$

$$\therefore D = 1, \qquad D = \frac{-1 \pm \sqrt{1-4}}{2}$$

$$\therefore D = 1, \qquad D = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$\therefore$ C.F. is

$$y = c_1 e^x + e^{-x/2}\left[c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right] \qquad \text{...(i)}$$

P.I. is

$$y = \frac{1}{D^3 - 1}(1 + 2e^x + e^{2x}) = \frac{1}{D^3 - 1}(e^{0 \cdot x} + 2e^x + e^{2x})$$

$$= -1 + 2\frac{1}{(D-1)(D^2+D+1)}e^x + \frac{1}{D^3 - 1}e^{2x}$$

$$= -1 + 2\frac{x}{3}e^x + \frac{1}{7}e^{2x} \qquad \text{...(ii)}$$

From (i), (ii); General Solution is y = C.F. + P.I.

Ex.90: Solve : $(D^3 - D^2 - D + 1)\,y = \cosh x \sin x$. **(Dec. 97, 5 M**

Sol. :

$$\text{A.E. is } D^3 - D^2 - D + 1 = 0$$

$$\therefore (D^2 - 1)(D - 1) = 0$$

$$\therefore (D - 1)^2 (D + 1) = 0$$

Roots are 1, 1, − 1.

∴ C.F. is

$$y = (c_1 + c_2 x)\,e^x + c_3 e^{-x}$$

P.I. is

$$y = \frac{1}{D^3 - D^2 - D + 1}(\cosh x \cdot \sin x) = \frac{1}{D^3 - D^2 - D + 1}\frac{1}{2}(e^x + e^{-x})\sin x$$

$$= \frac{1}{2}\frac{1}{D^3 - D^2 - D + 1}e^x \sin x + \frac{1}{2}\frac{1}{D^3 - D^2 - D + 1}e^{-x}\sin x$$

$$= \frac{1}{2}e^x \frac{1}{(D+1)^3 - (D+1)^2 - (D+1) + 1}\sin x$$

$$+ \frac{1}{2}e^{-x}\frac{1}{(D-1)^3 - (D-1)^2 - (D-1) + 1}\sin x$$

$$= \frac{e^x}{2} \frac{1}{D^3 + 2D^2} \sin x + \frac{e^{-x}}{2} \frac{1}{D^3 - 4D^2 + 4D} \sin x$$

$$= \frac{e^x}{2} \frac{1}{-D-2} \sin x + \frac{e^{-x}}{2} \frac{1}{3D+4} \sin x$$

$$= \left(\frac{e^x}{-2}\right) \frac{D-2}{D^2-4} \sin x + \frac{e^{-x}}{2} \frac{3D-4}{9D^2-16} \sin x$$

$$= -\frac{e^x}{2}\left(\frac{\cos x - 2\sin x}{-5}\right) + \frac{e^{-x}}{2}\left(\frac{3\cos x - 4\sin x}{-25}\right)$$

$$= \frac{e^x}{10}(\cos x - 2\sin x) - \frac{e^{-x}}{50}(3\cos x - 4\sin x)$$

General Solution is

$$y = c_1 e^{-x} + (c_2 + c_3 x) e^x + \frac{e^x}{10}(\cos x - 2\sin x) - \frac{e^{-x}}{50}(3\cos x - 4\sin x)$$

Q: $\frac{d^2y}{dx^2} - (a+b)\frac{dy}{dx} + aby = e^{ax} + e^{bx}$. (Dec.

A.:

$$\text{Let } D \equiv \frac{d}{dx}$$

$$\therefore \left[D^2 - (a+b)D + ab\right]y = e^{ax} + e^{bx}$$

$$\text{A.E. is } (D-a)(D-b) = 0$$

Roots are a, b

$\therefore$ C.F. is

$$y = c_1 e^{ax} + c_2 e^{bx}$$

P.I. is

$$y = \frac{1}{(D-a)(D-b)}\left(e^{ax} + e^{bx}\right)$$

$$= \frac{1}{(a-b)} x e^{ax} + \frac{1}{(b-a)} x e^{bx}$$

$$= \frac{x}{a-b}\left(e^{ax} - e^{bx}\right)$$

$\therefore$ General Solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 e^{ax} + c_2 e^{bx} + \frac{x}{(a-b)}\left(e^{ax} - e^{bx}\right)$$

25: $6\frac{d^2y}{dx^2} + 17\frac{dy}{dx} + 12y = e^{-3/2x} + 2^x$. **(Dec. 99)**

Sol. :

$$\text{Let} \quad D \equiv \frac{d}{dx}$$

$$\therefore \left[6D^2 + 17D + 12\right]y = e^{-\frac{3x}{2}} + 2^x$$

$$\text{A.E. is } 6D^2 + 17D + 12 = 0$$

$$(6D^2 + 9D + 8D + 12) = 0$$

$$\therefore (3D + 4)(2D + 3) = 0$$

$$\therefore \quad D = -\frac{4}{3}, -\frac{3}{2}$$

$\therefore$ C.F. is,

$$y = c_1 e^{-\frac{4x}{3}} + c_2 e^{-\frac{3x}{2}} \quad \text{...(i)}$$

P.I. is

$$y = \frac{1}{(6D^2 + 17D + 12)}\left(e^{-\frac{3x}{2}} + 2^x\right)$$

$$= \frac{1}{6D^2 + 17D + 12} e^{-\frac{3x}{2}} + \frac{1}{6D^2 + 17D + 12} e^{x \log 2}$$

$$= \frac{1}{(3D + 4)(2D + 3)} e^{-\frac{3x}{2}} + \frac{1}{6(\log 2)^2 + 17(\log 2) + 12} e^{x \log 2}$$

$$= \frac{1}{\left(-\frac{9}{2}+4\right)} x \cdot e^{-\frac{3x}{2}} + \frac{1}{6(\log 2)^2 + 17(\log 2) + 12} \cdot 2^x$$

$$= -2x\,e^{-\frac{3x}{2}} + \frac{1}{6(\log 2)^2 + 17(\log 2) + 12} \cdot 2^x \qquad \text{...(ii)}$$

General Solution is given by (i) and (ii),

i.e. $y = \text{C.F.} + \text{P.I.}$ is General Solution

36: $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = e^{-x}\log x + x^2.$ (Dec. 99, 16 Marks)

I. :

$$\text{Let} \quad D \equiv \frac{d}{dx}$$

$$\therefore \quad (D^2 + 2D + 1)\,y = e^{-x} \cdot \log x + x^2$$

$\therefore$ A.E. is

$$(D+1)^2 = 0$$

Roots are $-1, -1$

$\therefore$ C.F. is

$$y = (c_1 + c_2 x)\,e^{-x} \quad \text{...(i)}$$

P.I. is

$$y = \frac{1}{(D+1)^2}(e^{-x}\log x + x^2) = \frac{1}{(D+1)^2}e^{-x}\cdot\log x + \frac{1}{(D+1)^2}x^2$$

$$= e^{-x}\frac{1}{D^2}\log x + (1+D)^{-2}\cdot x^2$$

$$= e^{-x}\frac{1}{D}\left[\int \log x \cdot dx\right] + \left[1 + (-2)D + \frac{(-2)(-3)}{2}D^2\right]x^2$$

$$= e^{-x}\frac{1}{D}[x\log x - x] + [x^2 - 4x + 6]$$

$$= e^{-x}\left[\int x\log x\cdot dx - \int x\cdot dx\right] + (x^2 - 4x + 6)$$

$$= e^{-x}\left[\frac{x^2}{2}\log x - \int \frac{x^2}{2}\cdot\frac{1}{x}\cdot dx - \frac{x^2}{2}\right] + (x^2 - 4x + 6)$$

$$= e^{-x}\left[\frac{x^2}{2}\log x - \frac{x^2}{4} - \frac{x^2}{2}\right] + (x^2 - 4x + 6)$$

Ex. 30: $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = e^x \cos 2x + \cos 3x.$

(May 98, 5 Marks),(Dec

Sol. :

$$\text{Let} \quad D \equiv \frac{d}{dx},$$

$$(D^2 - 4D + 3)\,y = e^x \cos 2x + \cos 3x$$

$$\text{A.E. is } D^2 - 4D + 3 = 0$$

∴ Roots are 3, 1.

$$\therefore \quad \text{C.F. is } y = c_1 e^{3x} + c_2 e^x$$

P.I. is

$$y = \frac{1}{D^2 - 4D + 3}\left(e^x \cos 2x + \cos 3x\right)$$

$$= e^x \frac{1}{(D+1)^2 - 4(D+1) + 3} \cos 2x + \frac{1}{D^2 - 4D + 3} \cos 3x$$

$$= e^x \frac{1}{D^2 - 2D} \cos 2x + \frac{1}{-4D - 6} \cos 3x$$

$$= e^x \frac{1}{-4 - 2D} \cos 2x + \frac{1}{-4D - 6} \cos 3x$$

$$= \frac{e^x}{-2}\left(\frac{D-2}{D^2 - 4}\right) \cos 2x - \frac{1}{2}\frac{(2D - 3)}{(4D^2 - 9)} \cos 3x$$

$$= \frac{e^x}{-2}\left[\frac{-2 \sin 2x - 2 \cos 2x}{-8}\right] - \frac{1}{2}\frac{(-6 \sin 3x - 3 \cos 3x)}{-45}$$

$$= \frac{e^x}{-8}(\sin 2x + \cos 2x) - \frac{1}{30}(2 \sin 3x + \cos 3x)$$

∴ General Solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$\therefore \quad y = c_1 e^{3x} + c_2 e^x - \frac{e^x}{8}(\sin 2x + \cos 2x) - \frac{1}{30}(2 \sin 3x + \cos 3x)$$

Ex. 33 Solve $\frac{d^3y}{dx^3} - 4\frac{dy}{dx} = 2\cos h^2 2x.$ **(M.U. 1993, 94, 2**

: The auxiliary equation is $D^3 - 4D = 0$

$\therefore \quad D(D^2 - 4) = 0 \qquad \therefore \quad D = 0, 2, -2.$

$\therefore \quad$ C.F. is $y = c_1 + c_2e^{2x} + c_3e^{-2x}.$

$$\text{P.I.} = \frac{1}{D^3 - 4D} 2\cos h^2(2x) = \frac{1}{D^3 - 4D} 2\left(\frac{e^{2x} + e^{-2x}}{2}\right)^2$$

$$= \frac{1}{2} \cdot \frac{1}{D^3 - 4D}(e^{4x} + 2 + e^{-4x})$$

$$= \frac{1}{2}\left[\frac{1}{D^3 - 4D}e^{4x} + 2\frac{1}{D(D^2 - 4)}e^{0x} + \frac{1}{D^3 - 4D}e^{-4x}\right]$$

$$= \frac{1}{2}\left[\frac{1}{48}e^{4x} - \frac{x}{2} - \frac{1}{48}e^{-4x}\right]$$

$$= -\frac{x}{4} + \frac{1}{48}\left(\frac{e^{4x} - e^{-4x}}{2}\right)$$

$$= -\frac{x}{4} + \frac{1}{48}\sin h\, 4x.$$

$\therefore$ The complete solution is

$$y = c_1 + c_2e^{2x} + c_3e^{-2x} - \frac{x}{4} + \frac{1}{48}\sin h\, 4x.$$

(Q39)

$\therefore\ (D^2-1)(D^2+1)=0 \qquad \therefore\ D = 1, -1, +i, -i.$

$\therefore$ The C.F. is $y = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_3 \sin x.$

$$\text{P.I.} = \frac{1}{D^4-1}(e^x + \cos x \cos 3x)$$

$$= \frac{1}{D^4-1}\left[e^x + \frac{1}{2}(\cos 4x + \cos 2x)\right] \qquad \textbf{[Note this.]}$$

$$= \frac{1}{(D-1)(D+1)(D^2+1)}e^x + \frac{1}{2}\cdot\frac{1}{(D^4-1)}\cos 4x + \frac{1}{2(D^4-1)}\cos 2x$$

$$= \frac{x}{4}e^x + \frac{1}{510}\cos 4x + \frac{1}{30}\cos 2x.$$

Ex. Solve $(D-1)^2(D^2+1)y = e^x + \sin^2(x/2)$.

Sol. : The auxiliary equation is $(D-1)^2(D^2+1) = 0$

$\therefore \quad D = 1, 1, +i, -i.$

$\therefore$ The C.F. is $y = (c_1 + c_2 x)e^x + (c_3\cos x + c_4 \sin x)$.

$$\text{P.I.} = \frac{1}{(D-1)^2(D^2+1)}\left[e^x + \sin^2\frac{x}{2}\right]$$

Now, $$\frac{1}{(D-1)^2(D^2+1)}e^x = \frac{x^2}{2\,!}\cdot\frac{1}{2}e^x$$

and $$\frac{1}{(D-1)^2(D^2+1)}\sin^2\frac{x}{2} = \frac{1}{(D-1)^2(D^2+1)}\left[\frac{1-\cos x}{2}\right]$$

$$= \frac{1}{(D-1)^2(D^2+1)}\left(\frac{1}{2}e^{0x}\right) - \frac{1}{(D-1)^2(D^2+1)}\left(\frac{1}{2}\cos x\right)$$

$$= \frac{1}{(-1)^2(1)} \cdot \frac{1}{2} - \frac{1}{(D^2-2D+1)(D^2+1)}\left(\frac{1}{2}\cos x\right)$$

$$= \frac{1}{2} - \frac{1}{-2D} \cdot \frac{1}{(D^2+1)}\left(\frac{\cos x}{2}\right)$$

$$= \frac{1}{2} - \frac{1}{(D^2+1)} \cdot \frac{D}{(-2D^2)}\left(\frac{\cos x}{2}\right)$$

$$= \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{(D^2+1)} \frac{1}{(-1)}(-\sin x)$$

$$= \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{(D^2+1)}(\sin x)$$

$$= \frac{1}{2} + \frac{1}{4} \cdot \frac{x}{2D}\sin x \qquad \text{[By § 9 a]}$$

$$= \frac{1}{2} + \frac{x}{8}\int \sin x \, dx = \frac{1}{2} - \frac{x}{8}\cos x$$

(Or you can use formulae (1) of page 10.20 directly).

Hence, the complete solution is

$$y = (c_1 + c_2 x) e^x + c_3 \cos x + c_4 \sin x + \frac{1}{2} \cdot \frac{x^2}{2!} e^x + \frac{1}{2} - \frac{x}{8}\cos x.$$

Ex. 5 Solve $(D^2 + 2)\,y = e^x \cos x + x^2 e^{3x}$. **(M.U. 2**

: The auxiliary equation is $D^2 + 2 = 0$

$\therefore \quad D = +\sqrt{2} \cdot i, \ -\sqrt{2} \cdot i.$

$\therefore$ The C.F. is $y = c_1 \cos \sqrt{2} \cdot x + c_2 \sin \sqrt{2} \cdot x.$

$$\text{P.I.} = \frac{1}{D^2 + 2} e^x \cos x = e^x \cdot \frac{1}{(D+1)^2 + 2} \cos x$$

$$= e^x \cdot \frac{1}{D^2 + 2D + 3} \cdot \cos x = e^x \cdot \frac{1}{2D + 2} \cos x$$

$$= e^x \cdot \frac{1}{2} \cdot \frac{D-1}{D^2 - 1} \cdot \cos x = e^x \cdot \frac{1}{2} \cdot \frac{1}{-2} \cdot (-\sin x - \cos x)$$

$$= e^x \cdot \frac{1}{4} (\sin x + \cos x)$$

As in example 4 above P.I. corresponding to the second part

$$= \frac{e^{3x}}{11}\left(x^2 - \frac{12x}{11} + \frac{50}{121}\right)$$

$\therefore$ The complete solution is

$$y = c_1 \cos\sqrt{2}\cdot x + c_2 \sin\sqrt{2}\cdot x + e^x \cdot \frac{1}{4}(\sin x + \cos x)$$

$$+ \frac{e^{3x}}{11}\left(x^2 - \frac{12}{x} + \frac{50}{121}\right).$$

Ex. 8 : Solve $(D^3 - 7D - 6)$

Ex. 5 Solve $(D^3 - 7D - 6)\, y = \cos hx \cos x$. **(M.U. 2**

: The auxiliary equation is $D^3 - 7D - 6 = 0$.

$$\therefore \quad D^3 + D^2 - D^2 - D - 6D - 6 = 0$$

$$\therefore \quad (D+1)(D^2 - D - 6) = 0$$

$$\therefore \quad (D+1)(D+2)(D-3) = 0 \qquad \therefore \; D = -1, -2, 3.$$

$\therefore$ The C.F. is $y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{3x}$.

$$\text{P.I.} = \frac{1}{D^3 - 7D - 6} \cos hx \cos x = \frac{1}{D^3 - 7D - 6} \left(\frac{e^x + e^{-x}}{2} \right) \cdot \cos x$$

Now, $$\frac{1}{D^3 - 7D - 6} \cdot e^x \cos x = e^x \cdot \frac{1}{(D+1)^3 - 7(D+1) - 6} \cos x$$

$$= e^x \cdot \frac{1}{D^3 + 3D^2 - 4D - 12} \cos x$$

$$= e^x \cdot \frac{1}{-D - 3 - 4D - 12} \cos x \qquad (\text{Putting } D^2 = -1)$$

$$= -\frac{1}{5}\, e^x \cdot \frac{1}{D+3} \cos x = -\frac{1}{5} e^x \cdot \frac{(D-3)}{(D^2 - 9)} \cos x$$

$$= -\frac{1}{5}\, e^x \cdot \frac{1}{(-1-9)} \cdot (D-3) \cos x = \frac{e^x}{50} (-\sin x - 3\cos x)$$

Similarly, we find that

$$\frac{1}{D^3 - 7D - 6} \cdot e^{-x} \cos x = \frac{e^{-x}}{34} (3\cos x - 5\sin x)$$

$\therefore$ The complete solution is

$$y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{3x} - \frac{1}{100} \cdot e^x (\sin x + 3\cos x)$$

$$+ \frac{1}{68} \cdot e^{-x} (3\cos x - 5\sin x).$$

Ex. Solve $(D^2-4)\,y = x\sin hx$. **(M.U. 1991)**

: The auxiliary equation is $D^2-4=0 \quad \therefore \quad D = 2, -2$.

$\therefore$ The C.F. is $y = c_1e^{2x} + c_2e^{-2x}$.

$$\text{P.I.} = \frac{1}{D^2-4}\,x\sin hx = \frac{1}{D^2-4}\,x\cdot\left(\frac{e^x-e^{-x}}{2}\right)$$

$$= \frac{1}{2}\left[\frac{1}{D^2-4}\cdot xe^x - \frac{1}{D^2-4}\,x\cdot e^{-x}\right]$$

$$= \frac{1}{2}\left[x - \frac{1}{D^2-4}\cdot 2D\right]\frac{1}{D^2-4}\,e^x - \frac{1}{2}\left[x - \frac{1}{D^2-4}\cdot 2D\right]\frac{1}{D^2-4}\,e^{-x}$$

$$= \frac{1}{2}\left[x - \frac{1}{D^2-4}\cdot 2D\right]\left(-\frac{1}{3}e^x\right) - \frac{1}{2}\left[x - \frac{1}{D^2-4}\cdot 2D\right]\left(-\frac{1}{3}e^{-x}\right)$$

$$= -\frac{1}{6}\left[x \cdot e^{x} - \frac{1}{D^2 - 4} \cdot 2e^{x}\right] + \frac{1}{6}\left[x \cdot e^{-x} - \frac{1}{D^2 - 4} \cdot 2(-e^{-x})\right]$$

$$= -\frac{1}{6}\left[x \cdot e^{x} + \frac{2}{3} e^{x}\right] + \frac{1}{6}\left[x \cdot e^{-x} - \frac{2}{3} \cdot e^{-x}\right]$$

$$= -\frac{x}{6}\left(e^{x} - e^{-x}\right) - \frac{1}{6} \cdot \frac{2}{3}\left(e^{x} + e^{-x}\right)$$

$$= -\frac{x}{3}\left(\frac{e^{x} - e^{-x}}{2}\right) - \frac{2}{9}\left(\frac{e^{x} + e^{-x}}{2}\right) = \frac{x}{3} \sin hx - \frac{2}{9} \cos hx$$

∴ The complete solution is

$$y = c_1 e^{2x} + c_2 e^{-2x} - \frac{x}{3} \sin hx - \frac{2}{9} \cos hx.$$

(256)

$$\text{P.I.} = \frac{1}{D^2+4}(x\sin^2 x) = \frac{1}{D^2+4}x\left(\frac{1-\cos 2x}{2}\right)$$

$$= \frac{1}{2}\cdot\frac{1}{D^2+4}x - \frac{1}{2}\cdot\frac{1}{D^2+4}x\cos 2x.$$

Now, $\frac{1}{2}\cdot\frac{1}{D^2+4}x = \frac{1}{2}\cdot\frac{1}{4}\left(1-\frac{D^2}{4}\ldots\right)x = \frac{1}{8}x$

And $\frac{1}{2}\cdot\frac{1}{D^2+4}x\cos 2x = \frac{1}{2}\text{R. P. of }\frac{1}{D^2+4}x\cdot e^{2ix}$

$$= \frac{1}{2}\text{R. P. of } e^{2ix}\cdot\frac{1}{(D+2i)^2+4}x$$

$$= \frac{1}{2}\text{R. P. of } e^{2ix}\cdot\frac{1}{D^2+4iD}x$$

$$= \frac{1}{2}\text{ R. P. of } e^{2ix}\cdot\frac{1}{4iD}\cdot\frac{1}{[1+(D/4i)]}x$$

$$= \frac{1}{2}\text{R. P. of } e^{2ix}\cdot\frac{1}{4iD}\cdot\left[1-\frac{D}{4i}\ldots\right]x$$

$$= \frac{1}{2}\text{R. P. of } e^{2ix}\cdot\frac{1}{4iD}\cdot\left[x-\frac{1}{4i}\right]$$

$$= \frac{1}{2}\text{R. P. of } e^{2ix}\cdot\frac{1}{4i}\left[\frac{x^2}{2}-\frac{x}{4i}\right] \qquad \left[\because \frac{1}{D} = \int dx\right]$$

$$= \frac{1}{2}\text{R. P. of } e^{2ix}\left(\frac{x^2}{8i}+\frac{x}{16}\right)$$

$$= \frac{1}{2}\text{R. P. of } (\cos 2x + i\sin 2x)\left(\frac{x^2}{8i}+\frac{x}{16}\right)$$

$$= \frac{x}{32}\cos 2x + \frac{x^2}{16}\sin 2x$$

∴ The complete solution is

$$y = c_1\cos 2x + c_2\sin 2x + \frac{x}{8} - \frac{x}{32}\cos 2x - \frac{x^2}{16}\sin 2x.$$

Ex. 47. Solve : $(D^2 + 1)\, y = \tan x$ (May 98, 6 Marks)

Sol. :

Step I : C.F. is

$$y = c_1 \cos x + c_2 \sin x \quad \text{... (1)}$$

Let P. I. be $$y = u \cos x + v \sin x \quad \text{... (2)}$$

Step II : $$Dy = (u' \cos x + v' \sin x) + (-u \sin x + v \cos x)$$

Let $$u' \cos x + v' \sin x = 0 \quad \text{... (3)}$$

$$\therefore \quad Dy = (-u \sin x + v \cos x)$$

$$D^2 y = (-u' \sin x + v' \cos x) + (-u \cos x - v \sin x)$$

$$= (-u' \sin x + v' \cos x) - y$$

$$\therefore \quad (D^2 + 1)\, y = -u' \sin x + v' \cos x$$

Comparing with the given equation

$$-u' \sin x + v' \cos x = \tan x \quad \text{... (4)}$$

Solving (3) and (4), we get

$$u' = -\frac{\sin^2 x}{\cos x}, \qquad v' = \sin x$$

$$\therefore \quad u = -\int \frac{1 - \cos^2 x}{\cos x}\, dx = -\int \sec x\, dx + \int \cos x \cdot dx$$

$$\therefore \quad u = -\log(\sec x + \tan x) + \sin x, \quad v = -\cos x$$

$\therefore$ P. I. is $$y = -\cos x \cdot \log(\sec x + \tan x) + \sin x \cdot \cos x - \sin x \cdot \cos x$$

$$= -\cos x \cdot \log(\sec x + \tan x)$$

$\therefore$ G. S. is $$y = c_1 \cos x + c_2 \sin x - \cos x \log(\sec x + \tan x)$$

63 Use method of variation of parameters to solve the differential equation $y'' + 3y' + 2y = e^{e^x}$

(Dec. 95, Dec. 99, 5 Marks)

.:

Let $D \equiv \frac{d}{dx}$

$$\therefore \quad (D^2 + 3D + 2)\,y = e^{e^x}$$

A.E. is $D^2 + 3D + 2 = 0$

Roots are $-1, -2$

$$\therefore \text{ C.F. is } y = c_1 e^{-x} + c_2 e^{-2x} \qquad \text{...(i)}$$

Let P.I. be

$$y = u\,e^{-x} + v e^{-2x} \qquad \text{...(ii)}$$

where u, v are functions of x.

$$\therefore \quad Dy = (u' e^{-x} + v' e^{-2x}) + (-u\,e^{-x} - 2\,v e^{-2x}) \qquad \text{...(iii)}$$

Let $u' e^{-x} + v' e^{-2x} = 0$

$$\therefore \quad D^2 y = (-u' e^{-x} - 2v' e^{-2x}) + (u e^{-x} + 4v e^{-2x})$$

$$\therefore (D^2 + 3D + 2) y = -u' e^{-x} - 2v' e^{-2x}$$

Comparing with the given equation

$$-u' e^{-x} - 2v' e^{-2x} = e^{e^x} \qquad \text{...(iv)}$$

Taking (iii) + (iv); we get,

$$-v' e^{-2x} = e^{e^x}$$

$$\therefore \quad v' = -e^{2x} \cdot e^{e^x}$$

$$\therefore \quad v = -\int e^{2x} \cdot e^{e^x} \cdot dx$$

Let $e^x = t$

$$\therefore \quad e^x dx = dt = -\int t e^t \cdot dt = -(t e^t - e^t)$$

$$\therefore \quad v = -t e^t + e^t = e^{e^x} - e^x \cdot e^{e^x} \qquad \text{...(v)}$$

Taking 2 × (iii) + (iv), we get,

$$u' e^{-x} = e^{e^x}$$

$$\therefore \quad u' = e^x e^{e^x}$$

Integrating $\quad \therefore \quad u = \int e^x \cdot e^{e^x} \cdot dx$

Let $e^x = t = \int e^t \cdot dt = e^t = e^{e^x}$

∴ From (v) and (vi),

P.I. is

$$y = e^{e^x} \cdot e^{-x} + \left(e^{e^x} - e^x \cdot e^{e^x} \right) e^{-2x}$$

$$= e^{-2x} \cdot e^{e^x}$$

∴ G.S. is

$$y = \text{C.F.} + \text{P.I.}$$

$$\therefore \quad y = c_1 e^{-x} + c_2 e^{-2x} + e^{-2x} \cdot e^{e^x}$$

14. Solve : $\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$ by the method of variation of parameters. **(Dec. 97, 6 Marks)**

Sol. :

$$\text{Let} \quad D \equiv \frac{d}{dx},$$

$$\therefore \quad (D^2 - 1)y = \frac{2}{1+e^x}$$

$$\text{A.E. is } D^2 - 1 = 0$$

Roots are ± 1

$$\therefore \text{ C.F. is } y = c_1 e^x + c_2 e^{-x} \quad \text{...(i)}$$

Let P.I. be

$$y = u e^x + v e^{-x} \quad \text{...(ii)}$$

$$\therefore \quad Dy = (u' e^x + v' e^{-x}) + (u e^x - v e^{-x})$$

$$\text{Let} \quad u' e^x + v' e^{-x} = 0 \quad \text{...(iii)}$$

$$\therefore \quad D^2 y = (u' e^x - v' e^{-x}) + (u e^x + v e^{-x})$$

$$\therefore \quad (D^2 - 1)y = u' e^x - v' e^{-x}$$

$$\therefore \quad u' e^x - v' e^{-x} = \frac{2}{1+e^x} \quad \text{...(iv)}$$

Adding (iii) and (iv),

$$u' = \frac{1}{e^x(1+e^x)} \qquad \therefore \quad u = \int \frac{1}{e^x(e^x+1)}\,dx$$

$$= \int \frac{1}{\left(1+\frac{1}{e^{-x}}\right)} e^{-x} \cdot dx = \int \frac{e^{-x}}{(e^{-x}+1)} e^{-x} \cdot dx$$

Let $e^{-x} + 1 = u$

$$\therefore \; -e^{-x} \cdot dx = du = \int \frac{(u-1)}{u} (-du) = -\int du + \int \frac{1}{u} du = -u + \log u$$

$$= -e^{-x} - 1 + \log(e^{-x}+1)$$

Taking (iii) – (iv); we get,

$$2v' e^{-x} = \frac{-2}{1+e^{x}}$$

$$v' = \frac{-e^{x}}{e^{x}+1}$$

$$\therefore v = -\log(e^{x}+1)$$

Substituting (v) and (vi) in (ii), we get,

$$\text{P.I. } y = \left[-e^{-x} - 1 + \log(e^{-x}+1)\right] e^{x} - e^{-x} \log(e^{x}+1)$$

$$\therefore \quad y = -1 - e^{x} + e^{x} \log(e^{-x}+1) - e^{-x} \log(e^{x}+1)$$

$\therefore$ General Solution is

$$y = \text{C.F.} + \text{P.I.}$$

/85: Using method of variation of parameters.
Solve the equation :

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = \frac{e^{3x}}{x^2}$$

(

ol. :

$$\text{Let } D \equiv \frac{d}{dx}$$

$$\therefore \quad (D^2 - 6D + 9)\,y = \frac{e^{3x}}{x^2}$$

A.E. is

$$D^2 - 6D + 9 = 0$$

$$\therefore \quad (D-3)^2 = 0$$

∴ Roots are 3, 3.

∴ C.F. is

$$y = (c_1 + c_2 x)\,e^{3x}$$

Let P.I. be

$$y = (u + vx)\,e^{3x}$$

$$\therefore \quad y = u\,e^{3x} + v\,(x\,e^{3x})$$

$$\therefore \quad Dy = \left[u'\,e^{3x} + v'\,(x\,e^{3x})\right] + \left[3\,u\,e^{3x} + v\left(e^{3x} + 3x\,e^{3x}\right)\right]$$

Let $u' e^{3x} + v' (x e^{3x}) = 0$

$$\therefore \quad u' + x v' = 0 \qquad ...(iii)$$

$$\therefore \quad Dy = 3 u e^{3x} + v\left(e^{3x} + 3 x e^{3x} \right)$$

$$\therefore \quad D^2 y = \left[3 u' e^{3x} + v'\left(e^{3x} + 3x e^{3x} \right) \right] + u [9 e^{3x}] + v\left(3 e^{3x} + 3 e^{3x} + 9 x e^{3x} \right)$$

$$\therefore \quad (D^2 - 6D + 9) y = 3 u' e^{3x} + v' (e^{3x} + 3 x e^{3x})$$

Comparing with the given equation,

$$3 u' e^{3x} + v' e^{3x} (1 + 3x) = \frac{e^{3x}}{x^2}$$

$$\therefore \quad 3 u' + v' (1 + 3x) = \frac{1}{x^2}$$

$$\therefore \quad 3 (u' + x v') + v' = \frac{1}{x^2}$$

From (iii),

$$3 (0) + v' = \frac{1}{x^2}$$

$$\therefore \quad v' = \frac{1}{x^2}$$

$$\therefore \quad v = -\frac{1}{x} \qquad ...(iv)$$

Now, from (iii),

$$u' = - x v' = -\frac{1}{x} \qquad \therefore \quad u = - \log x \qquad ...(v)$$

$\therefore$ from (ii), P.I. is

$$y = [- \log x - 1] e^{3x} = - (1 + \log x) e^{3x}$$

$\therefore$ General Solution is $y = C.F. + P.I.$

$$\text{i.e.} \quad y = (c_1 + c_2 x) e^{3x} - (1 + \log x) e^{3x}$$

$$= (c_1 - 1) e^{3x} + c_2 x e^{3x} - \log x \cdot e^{3x}$$

i.e. General Solution is,

$$y = c_1' e^{3x} + c_2 x e^{3x} - \log x \cdot e^{3x}$$

Ex. 157 Solve : $(D^2 + D) y = 5 e^x - \sin 2x$ using method of undetermined coefficients.

(Dec. 9

Sol. :

$$\text{A.E. is } D^2 + D = 0$$

Roots are 0, – 1.

$\therefore$ C.F. is $\quad y = c_1 + c_2 e^{-x}$

Let P.I. be

$$y = A e^x + B \sin 2x + C \cos 2x$$

$$\therefore \quad Dy = A e^x + 2B \cos 2x - 2C \sin 2x$$

$$D^2 y = A e^x - 4B \sin 2x - 4C \cos 2x$$

Substituting in the given equation,

$$(A e^x - 4B \sin 2x - 4C \cos 2x) + (A e^x + 2B \cos 2x - 2C \sin 2x) = 5 e^x - \sin 2x$$

$$\therefore 2 A e^x - \sin 2x (4B + 2C) + \cos 2x (2B - 4C) = 5 e^x - \sin 2x$$

Comparing coefficients,

$$2A = 5 \qquad \therefore A = 5/2$$

$$4B + 2C = 1$$

$$\text{and} \quad 2B - 4C = 0$$

Solving,

$$B = \frac{1}{5}, C = \frac{1}{10}$$

$\therefore$ P.I. is

$$y = \frac{5}{2}e^{x} + \frac{1}{5}\sin 2x + \frac{1}{10}\cos 2x$$

$\therefore$ General Solution is

$$y = c_1 + c_2 e^{-x} + \frac{5}{2}e^{x} + \frac{1}{5}\sin 2x + \frac{1}{10}\cos 2x$$

39: Solve : $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = x^2 + 1$. Use method of undetermined

:

$$\text{Let } D \equiv \frac{d}{dx}$$

$$\therefore \quad (D^2 - 3D + 2)y = x^2 + 1$$

$$\therefore \quad \text{A.E. is } D^2 - 3D + 2 = 0$$

Roots are 1, 2

$\therefore$ C.F. is

$$y = c_1 e^x + c_2 e^{2x}$$

Let P.I. be

$$y = Ax^2 + Bx + C$$

$$\therefore \quad Dy = 2Ax + B$$

$$D^2 y = 2A$$

Substituting in the given equation,

$$(2A) - 3(2Ax + B) + 2(Ax^2 + Bx + C) = x^2 + 1$$

Equating coefficients, we get,

$$A = \frac{1}{2}, \qquad B = \frac{3}{2}, \qquad C = \frac{9}{4}$$

$\therefore$ P.I is

$$y = \frac{x^2}{2} + \frac{3x}{2} + \frac{9}{4}$$

From (i) and (iii),

General Solution is,

$$y = C.F. + P.I.$$

$$\therefore \quad y = c_1 e^x + c_2 e^{2x} + \frac{x^2}{2} + \frac{3x}{2} + \frac{9}{4}.$$

(870)

$\therefore$ C.F. is $$y = c_1 e^{-2x} + c_2 e^{-x}$$

Let P.I. be

$$y = A e^{3x} + B x^2 + Cx + D$$

$$\therefore \quad Dy = 3 A e^{3x} + 2 Bx + C$$

$$D^2 y = 9 A e^{3x} + 2B$$

Substituting in the given equation,

$$\left[9 A e^{3x} + 2B \right] + 3\left[3A e^{3x} + 2 Bx + C \right] + 2\left[A e^{3x} + Bx^2 + Cx + D \right] = 10 e^{3x} + 4 x^2$$

$$\therefore (20 A) e^{3x} + 2 B x^2 + (6B + 2C) x + 2B + 3C + 2D = 10 e^{3x} + 4 x^2$$

Comparing coefficients.

$$20 A = 10 \qquad \therefore \quad A = \frac{1}{2}$$

$$2B = 4 \qquad \therefore \quad B = 2$$

$$6B + 2C = 0 \qquad \therefore \quad C = -6$$

$$2B + 3C + 2D = 0 \qquad \therefore \quad D = -7$$

$\therefore$ P.I. is

$$y = \frac{1}{2} e^{3x} + 2 x^2 - 6x - 7$$

$\therefore$ From (i) and (iii),

General Solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 e^{-2x} + c_2 e^{-x} + \frac{1}{2} e^{3x} + 2 x^2 - 6x - 7$$

Ex. 5: Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 3y = x^2 + \sin 2x$.

: The auxiliary equation is $D^2 + 2D + 3 = 0$

$$\therefore \quad D = \frac{2 \pm 2\sqrt{2}\, i}{2} = 1 \pm \sqrt{2}\, i$$

$\therefore$ The C.F. is $y = e^x(c_1 \cos\sqrt{2}\, x + c_2 \sin\sqrt{2}\, x)$.

Let the P.I. be $y = Ax^2 + Bx + C + P\cos 2x + Q\sin 2x$.

$$\therefore \quad Dy = 2Ax + B - 2P\sin 2x + 2Q\cos 2x$$

$$D^2 y = 2A - 4P\cos 2x - 4Q\sin 2x.$$

Putting these values in the given equation.

$$2A - 4P\cos 2x - 4Q\sin 2x - 4Ax - 2B$$
$$+ 4P\sin 2x - 4Q\cos 2x + 3Ax^2 + 3Bx$$
$$+ 3C + 3P\cos 2x + 3Q\sin 2x = x^2 + \sin 2x.$$

$$\therefore \quad 3Ax^2 + (3B - 4A)\, x + (2A - 2B + 3C)$$
$$+ (4P - Q)\sin 2x + (-P - 4Q) = x^2 + \sin 2x.$$

Equating the coefficients.

$3A = 1 \quad \therefore \; A = \frac{1}{3} \; ; \; 3B - 4A = 0 \quad \therefore \; 3B = \frac{4}{3} \quad \therefore \; B = \frac{4}{9} \; ;$

$2A - 2B + 3C = 0 \quad \therefore \; \frac{2}{3} - \frac{8}{9} + 3C = 0 \quad \therefore \; 3C = \frac{2}{9} \quad \therefore \; c = \frac{2}{27}$

$4P - Q = 1 \;\text{ and }\; P + 4Q = 0 \quad \therefore \; P = -4Q$

$\therefore \; -16Q - Q = 1 \quad \therefore \; Q = \frac{-1}{17}, \; P = +\frac{4}{17}$

$\therefore \; \text{P.I.} = \frac{1}{3}x^2 + \frac{4}{9}x + \frac{2}{27} + \frac{4}{17}\cos 2x - \frac{1}{17}\sin 2x$

$\therefore$ The complete solution is

$$y = e^x(c_1 \cos\sqrt{2}\,x + c_2 \sin\sqrt{2}\,x) + \frac{1}{3}x^2 + \frac{4}{9}x + \frac{2}{17} + \frac{4}{17}\cos 2x - \frac{1}{17}\sin 2x.$$

Ex. 5: Solve $(D^2 - D)\,y = e^x \sin x$. (M

: The auxiliary equation is $D(D-1) = 0$ $\quad \therefore \; D = 0, 1.$

$\therefore$ The C.F. is $y = c_1 + c_2 e^x$.

Assume P.I. as $y = Ae^x \sin x + Be^x \cos x$.

$$\therefore \quad Dy = Ae^x \sin x + Ae^x \cos x + Be^x \cos x - Be^x \sin x$$

$$= (A - B)\,e^x \sin x + (A + B)\,e^x \cos x$$

$$D^2 y = (A - B)\,e^x \sin x + (A - B)\,e^x \cos x$$

$$+ (A + B)\,e^x \cos x - (A + B)\,e^x \sin x$$

$$= -2Be^x \sin x + 2Ae^x \cos x$$

Putting these in the given equation, we get

$$-2\,Be^x \sin x + 2\,Ae^x \cos x - (A - B)\,e^x \sin x$$

$$- (A - B)\,e^x \cos x = e^x \sin x$$

$$\therefore \quad (-A - B)\,e^x \sin x + (A - B)\,e^x \cos x = e^x \sin x$$

Equating the coefficients

$$\therefore \quad -A - B = 1 \text{ and } A - B = 0 \quad \therefore \; A = B = -\frac{1}{2}$$

$\therefore$ The complete solution is

$$y = c_1 + c_2\,x - \frac{1}{2} e^x \sin x - \frac{1}{2} e^x \cos x.$$

15) Solve : $(x^2 D^2 - 3xD + 1)\, y = \dfrac{\sin(\log x)}{x}$

A :

Let $z = \log x$ or $x = e^z$

Let $\theta \equiv \dfrac{d}{dz}$; and using $x^r D^r \equiv \theta(\theta - 1)\ldots(\theta - r + 1)$, we get

$$[\theta(\theta - 1) - 3\theta + 1]\, y = \frac{\sin z}{e^z} \quad \text{i.e.} \quad (\theta^2 - 4\theta + 1)\, y = e^{-z}\sin z$$

is a linear equation with constant coefficients.

A.E. is $\theta^2 - 4\theta + 1 = 0$

Roots are $2 \pm \sqrt{3}$

(Note : Roots are not complex numbers)

$$\therefore \text{ C.F. is } y = c_1 e^{(2+\sqrt{3})z} + c_2 e^{(2-\sqrt{3})z} \quad \ldots (1)$$

$$\text{P.I. is } y = \frac{1}{\theta^2 - 4\theta + 1} e^{-z}\sin z$$

$$= e^{-z}\frac{1}{(\theta - 1)^2 - 4(\theta - 1) + 1}\sin z = e^{-z}\frac{1}{\theta^2 - 6\theta + 6}\sin z$$

$$= e^{-z}\frac{1}{5 - 6\theta}\sin z \text{ (replace } \theta^2 \text{ by } -1\text{)} = e^{-z}\frac{(5 + 6\theta)}{25 - 36\theta^2}\sin z = \frac{e^{-z}}{61}(5\sin z + 6\cos z)$$

$$\therefore \quad \text{G.S. is } y = e^{2z}[c_1 e^{\sqrt{3}z} + c_2 e^{-\sqrt{3}z}] + \frac{e^{-z}}{61}(5\sin z + 6\cos z)$$

Replacing z by log x, we get

$$y = x^2[c_1 x^{\sqrt{3}} + c_2 x^{-\sqrt{3}}] + \frac{1}{61x}[5\sin(\log x) + 6\cos(\log x)]$$

(76) $x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} + 2y = 10\left(x + \frac{1}{x}\right)$ **(May 95)**

: The equation is a homogeneous linear equation.

Let $D = \frac{d}{dx}$; hence the equation becomes

$$\left(x^3 D^3 + 2x^2 D^2 + 2\right) y = 10\left(x + \frac{1}{x}\right)$$

Let $z = \log x,\ x = e^z$,

let $\theta = \frac{d}{dz}$ and

$$x^r D^r \equiv \theta(\theta - 1)\ldots\ldots(\theta - r + 1)$$

$$[\theta(\theta-1)(\theta-2) + 2\theta(\theta-1) + 2]\, y = 10\,(e^z + e^{-z})$$

$$\therefore\quad \left[\theta^3 - \theta^2 + 2\right] y = 10\,(e^z + e^{-z})$$

is a linear equation with constant coefficients.

A.E. is $\theta^3 - \theta^2 + 2 = 0$

$$\therefore\quad (\theta + 1)(\theta^2 - 2\theta + 2) = 0$$

$$\therefore\quad Q = -1,\ 1 \pm i$$

$\therefore$ C.F. is

$$y = c_1 e^{-z} + e^z\,(c_2 \cos z + c_3 \sin z) \qquad \ldots\text{(i)}$$

and P.I. is

$$y = 10\,\frac{1}{(\theta+1)(\theta^2 - 2\theta + 2)}\,(e^z + e^{-z})$$

$$= 10\left[\frac{1}{2}e^z + \frac{z}{5}\cdot e^{-z}\right] \qquad \ldots\ldots\text{(ii)}$$

ı (i) and (ii), General Solution is,

$$y = c_1 e^{-z} + e^z\,(c_2 \cos z + c_3 \sin z) + 10\left[\frac{1}{2}e^z + \frac{z\,e^{-z}}{5}\right]$$

acing z by log x, we get,

$$y = \frac{c_1}{x} + x\left[c_2 \cos(\log x) + c_3 \sin(\log x)\right] + 5x + 2\left(\frac{\log x}{x}\right)$$

is the General Solution.

(7) Solve the equation,

$$x^2 \frac{d^3y}{dx^3} + 2x \frac{d^2y}{dx^2} + 3\frac{dy}{dx} = x + 1.$$

: The equation is

$$x^2 \frac{d^3y}{dx^3} + 2x \frac{d^2y}{dx^2} + 3\frac{dy}{dx} = x + 1$$

$$\therefore \quad x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} + 3x\frac{dy}{dx} = x^2 + x$$

homogeneous equation.

$$\text{Let} \quad D \equiv \frac{d}{dx}$$

$$[x^3 D^3 + 2x^2 D^2 + 3xD]y = x^2 + x$$

$$\text{Let} \quad z = \log x, \; x = e^z$$

$$\text{Let} \quad \theta \equiv \frac{d}{dz}$$

$$\text{and} \quad x^r D^r \equiv \theta(\theta - 1) \ldots (\theta - r + 1)$$

$$[\theta(\theta-1)(\theta-2) + 2\theta(\theta-1) + 3\theta]y = e^{2z} + e^z$$

$$\therefore \quad [\theta^3 - \theta^2 + 3\theta]y = e^{2z} + e^z$$

is a linear equation with constant coefficients.

A.E. is $\theta^3 - \theta^2 + 3\theta = 0$

i.e. $\theta(\theta^2 - \theta + 3) = 0$

The roots are $\theta = 0; \dfrac{1 \pm \sqrt{1-12}}{2}$

i.e., $\theta = 0; \dfrac{1}{2} \pm i\dfrac{\sqrt{11}}{2}$

$\therefore$ C.F. is

$$y = c_1 + e^{z/2}\left[c_2 \cos\frac{\sqrt{11}}{2}z + c_3 \sin\frac{\sqrt{11}}{2}z\right]$$

and P.I. is

$$y = \frac{1}{\theta(\theta^2 - \theta + 3)}(e^{2z} + e^z)$$

$$= \frac{1}{2(5)}e^{2z} + \frac{1}{3}e^z$$

$$= \frac{1}{10}e^{2z} + \frac{1}{3}e^z$$

From (i), (ii), G.S. is

$$y = c_1 + e^{z/2}\left[c_2 \cos\frac{\sqrt{11}}{2}z + c_3 \sin\frac{\sqrt{11}}{2}z\right] + \frac{1}{10}e^{2z} + \frac{1}{3}e^z$$

replacing e^z by x,

$$y = c_1 + \sqrt{x}\left[c_2 \cos\left(\frac{\sqrt{11}}{2}\log x\right) + c_3 \sin\left(\frac{\sqrt{11}}{2}\log x\right)\right] + \frac{1}{10}x^2 + \frac{1}{3}x$$

Q. 78 Solve : $x^2 \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^2 + 2\log x.$

Sol. :

The equation is,

$$x^2 \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^2 + 2\log x$$

The equation is homogeneous.

$$\text{Let, } D \equiv \frac{d}{dx}$$

$$\therefore \quad [x^2D^2 - 2xD - 4]y = x^2 + 2\log x$$

$$\text{Let } z = \log x,\ x = e^z,\ \theta \equiv \frac{d}{dz},$$

$$x^r D^r \equiv \theta(\theta - 1)\ldots(\theta - r + 1)$$

$$\therefore \quad [\theta(\theta - 1) - 2\theta - 4]y = e^{2z} + 2z$$

$\therefore \ (\theta^2 - 3\theta - 4)\,y = e^{2z} + 2z$

ar differential equation with constant coefficients.

A.E. is $\theta^2 - 3\theta - 4 = 0$

$(\theta - 4)(\theta + 1) = 0$

ts are 4, − 1

$\therefore$ C.F. is $y = c_1 e^{4z} + c_2 e^{-z}$ ...(i)

. is $y = \dfrac{1}{\theta^2 - 3\theta - 4}\,(e^{2z} + 2z)$

$$\therefore \quad y = \frac{1}{\theta^2 - 3\theta - 4}\,e^{2z} + 2\,\frac{1}{\theta^2 - 3\theta - 4}\,z$$

$$= -\frac{1}{6}e^{2z} + \frac{2}{-4}\left[1 - \left(\frac{\theta^2 - 3\theta}{4}\right)\right]^{-1} z$$

$$= -\frac{1}{6}e^{2z} - \frac{1}{2}\left[1 + \left(\frac{\theta^2 - 3\theta}{4}\right)\right] z = -\frac{1}{6}\,e^{2z} - \frac{1}{2}\left[z - \frac{3}{4}\right]$$

$$= -\frac{1}{6}e^{2z} - \frac{z}{2} + \frac{3}{8} \quad \text{...(ii)}$$

i) and (ii), General Solution is

$$y = c_1 e^{4z} + c_2 e^{-z} - \frac{1}{6}e^{2z} - \frac{z}{2} + \frac{3}{8}$$

ing e^z by x and z by log x, we get,

$$y = c_1 x^4 + \frac{c_2}{x} - \frac{x^2}{6} - \frac{\log x}{2} + \frac{3}{8}$$ is General Solution.

Solve the equation,

$$x^2 \frac{d^2 y}{dx^2} + 2x \frac{dy}{dx} - 20\,y = (x+1)^2.$$

ol. : The equation is,

$$x^2 \frac{d^2 y}{dx^2} + 2x \frac{dy}{dx} - 20\,y = (x+1)^2$$

he equation is homogeneous.

$$\text{Let} \quad D \equiv \frac{d}{dx}$$

$$[\,x^2 D^2 + 2\,xD - 20\,]\,y = (x+1)^2$$

$$\text{Let} \quad z = \log x,\ x = e^z,\ \theta \equiv \frac{d}{dz},$$

$$\text{and } x^r D^r \equiv \theta(\theta-1)\ldots(\theta-r+1)$$

$$\therefore \quad [\theta(\theta-1)+2\theta-20]y = (1+e^z)^2$$

$$\therefore \quad [\theta^2+\theta-20]y = 1+2e^z+e^{2z}$$

$$\text{A.E. is } \theta^2+\theta-20 = 0$$

$$\therefore \quad \theta = -5, 4$$

$$\therefore \text{ C.F. is } y = c_1 e^{-5z} + c_2 e^{4z}$$

and P.I. is

$$y = \frac{1}{\theta^2+\theta-20}(1+2e^z+e^{2z}) = \frac{1}{\theta^2+\theta-20}[e^{0\cdot z}+2e^z+e^{2z}]$$

$$= \frac{1}{-20}e^{0.z} + 2\frac{1}{-17}e^z + \frac{1}{-14}e^{2z} = -\frac{1}{20} - \frac{2}{17}e^z - \frac{1}{14}e^{2z}$$

$\therefore$ G.S. is

$$y = c_1 e^{-5z} + c_2 e^{4z} - \frac{1}{20} - \frac{2}{17}e^z - \frac{1}{14}e^{2z}$$

Replacing e^z by x; we get,

$$y = \frac{c_1}{x^5} + c_2 x^4 - \frac{1}{20} - \frac{2}{17}x - \frac{1}{14}x^2 \quad \text{is General Solution}$$

4 : For steady heat flow through the wall of a hollow sphere of inner and outer radii r_1 and r_2 respectively, the temperature u at a distance r (where $r_1 \le r \le r_2$) from the centre of the sphere is given by $r\frac{d^2u}{dr^2}+2\frac{du}{dr} = 0$, If u_1 and u_2 are temperatures at inner and outer surfaces of the sphere find u in terms of r. **(May 98, 5 Marks) (Dec. 98, 6 Marks)**

(80)

Sol. : The given equation can be written as

$$r^2\frac{d^2u}{dr^2}+2r\frac{du}{dr} = 0$$

is a homogeneous equation

$$\text{Let} \quad D \equiv \frac{d}{dr}$$

$$\therefore \quad [r^2D^2+2rD]u = 0$$

$$\text{Let} \quad z = \log r,\ r = e^z;\ \theta \equiv \frac{d}{dz}$$

$$\therefore \quad rD \equiv \theta,\ r^2D^2 \equiv \theta(\theta-1)$$

The equation becomes

$$[\theta(\theta-1)+2\theta]u = 0$$

$$\therefore \quad (\theta^2+\theta)u = 0$$

∴ G.S. is

$$u = c_1 + c_2 e^{-z}$$

$$\therefore \quad u = c_1 + \frac{c_2}{r} \qquad ...(1)$$

We impose the given conditions :

$r = r_1$ when $u = u_1$

$$\therefore \quad u_1 = c_1 + \frac{c_2}{r_1} \qquad ...(2)$$

$r = r_2$ when $u = u_2$

$$\therefore \quad u_2 = c_1 + \frac{c_2}{r_2} \qquad ...(3)$$

Taking (2) – (3); we get,

$$u_1 - u_2 = c_2\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

$$\therefore \quad u_1 - u_2 = c_2\left(\frac{r_2 - r_1}{r_1 r_2}\right)$$

$$\therefore \quad c_2 = \frac{r_1 r_2 (u_1 - u_2)}{(r_2 - r_1)}$$

Taking (2) × r_1 – (3) × r_2, we get,

$$r_1 u_1 - r_2 u_2 = r_1 c_1 - r_2 c_1 = (r_1 - r_2) c_1$$

$$\therefore \quad c_1 = \frac{r_1 u_1 - r_2 u_2}{r_1 - r_2}$$

∴ from (1),

$$u = \left(\frac{r_2 u_2 - r_1 u_1}{r_2 - r_1}\right) + \frac{r_1 r_2 (u_1 - u_2)}{r(r_2 - r_1)}$$

Ex. 8 | Solve $x^2 \frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 5y = \sin(\log x)$. **(M.U. 1991, 92)**

Putting $z = \log x$ and $x = e^z$, we get

$$[D(D-1) - 3D + 5]\,y = \sin z \qquad \therefore\ (D^2 - 4D + 5)\,y = \sin z$$

$\therefore$ The A.E. is $(D^2 - 4D + 5) = 0$.

$$\therefore\quad D = \frac{4 \pm 2i}{2} = 2 \pm i$$

$\therefore$ The C.F. is $y = e^{2z}(c_1 \cos z + c_2 \sin z)$.

$$\text{P.I.} = \frac{1}{D^2 - 4D + 5}\sin z = \frac{1}{-4D + 4}\cdot\sin z \qquad \text{[By § 9 of Chapter 10]}$$

$$= \frac{1}{-4}\cdot\frac{D+1}{D^2-1}\cdot\sin z = \frac{1}{8}(D+1)\sin z$$

$$= \frac{1}{8}(\cos z + \sin z)$$

$\therefore$ The complete solution is

$$y = e^{2z}(c_1 \cos z + c_2 \sin z) + \frac{1}{8}(\cos z + \sin z)$$

Resubstituting in terms of x, we get,

$$y = x^2(c_1 \cos\log x + c_2 \sin\log x) + \frac{1}{8}(\cos\log x + \sin\log x).$$

Ex. 82 Solve $x^3 \frac{d^3y}{dx^3} - x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} - 2y = x^3 + 3x.$ (M.U. 199

: Putting $z = \log x$ and $x = e^z$, we get

$$[D(D-1)(D-2) - D(D-1) + 2D - 2]\, y = e^{3z} + 3e^z$$

$\therefore (D^3 - 3D^2 + 2D - D^2 + D + 2D - 2)y = e^{3z} + 3e^z$

$\therefore (D^3 - 4D^2 + 5D - 2)\, y = e^{3z} + 3e^z$

$\therefore$ The A.E. is $D^3 - 4D^2 + 5D - 2 = 0.$

$\therefore D^3 - D^2 - 3D^2 + 3D + 2D - 2 = 0$

$\therefore (D-1)(D^2 - 3D + 2) = 0 \quad \therefore (D-1)(D-1)(D-2) = 0$

$\therefore D = 1, 1, 2.$

$\therefore$ The C.F. is $y = (c_1 + c_2 z)\, e^z + c_3 e^{2z}.$

$$\therefore \quad \text{P.I.} = \frac{1}{(D-1)^2(D-2)}e^{3z} + \frac{1}{(D-1)^2(D-2)}3e^{z}$$

$$= \frac{1}{(3-1)^2(3-2)}e^{3z} + \frac{z^2}{2}\cdot\frac{1}{(1-2)}3e^{z}$$

$$= \frac{e^{3z}}{4} - \frac{z^2}{2}3e^{z}$$

∴ The complete solution is

$$y = (c_1 + c_2 z)e^{z} + c_3 e^{2z} + \frac{e^{3z}}{4} - \frac{3z^2}{2}e^{z}$$

$$\therefore \quad y = (c_1 + c_2 \log x)x + c_3 x^2 + \frac{x^3}{4} - \frac{3x}{2}(\log x)^2.$$

Ex. 83 Solve. $x^2 \frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 5y = x^2 \sin \log x.$

As seen in the example 2 above

$$C.F. = e^{2z}(c_1 \cos z + c_2 \sin z)$$

$$P.I. = \frac{1}{D^2 - 4D + 5} e^{2z} \cdot \sin z$$

$$= e^{2z} \cdot \frac{1}{(D+2)^2 - 4(D+2) + 5} \sin z$$

$$= e^{2z} \cdot \frac{1}{D^2 + 1} \sin z$$

$$= e^{2z} \cdot \left(\frac{-z}{2}\right) \cos z$$ [By § 9(a) of Cl

The complete solution is

$$y = e^{2z}(c_1 \cos z + c_2 \sin z) - \frac{1}{2} e^{2z} \cdot z \cos z$$

$$\therefore \quad y = x^2 (c_1 \cos \log x + c_2 \sin \log x) - \frac{1}{2} x^2 \log x \cos \log x.$$

x. 8) Solve $x^2 \frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + y = \frac{(\sin\log x)+1}{x}$. **(M.U. 2002)**

utting $z = \log x$ and $x = e^z$, we get

$$[D(D-1) - 3D + 1]\,y = (\sin z + 1)\cdot e^{-z}$$

$$\therefore (D^2 - 4D + 1)\,y = e^{-z}\sin z + e^{-z}$$

$\therefore$ The A.E. is $D^2 - 4D + 1 = 0 \quad \therefore D = 2 \pm \sqrt{3}$.

$\therefore$ The C.F. is $y = Ae^{(2+\sqrt{3})z} + Be^{(2-\sqrt{3})z}$.

$\therefore$ $y = e^{2z}(Ae^{\sqrt{3}\cdot z} + Be^{-\sqrt{3}\cdot z})$ which can be expressed as

$$y = e^{2z}(c_1\cos h\sqrt{3}\cdot z + c_2\sin h\sqrt{3}\cdot z)$$

.I. for $e^{-z} = \frac{1}{D^2 - 4D + 1}e^{-z} = \frac{1}{6}e^{-z}$

.I. for $e^{-z}\sin z = e^{-z}\cdot\frac{1}{(D-1)^2 - 4(D-1) + 1}\sin z$

$$= e^{-z}\cdot\frac{1}{D^2 - 6D + 6}\sin z = e^{-z}\cdot\frac{1}{5 - 6D}\sin z$$

$$= e^{-z}\cdot\frac{5 + 6D}{25 - 36D^2}\sin z = e^{-z}\frac{(5\sin z + 6\cos z)}{61}$$

. The complete solution is

$$y = e^{2z}(c_1\cos h\sqrt{3}\cdot z + c_2\sin h\sqrt{3}\cdot z) + \frac{1}{6}e^{-z} + \frac{e^{-z}}{61}(5\sin z + 6\cos z)$$

$$\therefore \quad y = x^2[c_1\cos h(\sqrt{3}\log x) + c_2\sin h(\sqrt{3}\log x)] + \frac{1}{6x} + \frac{1}{61x}[5\sin(\log x) + 6\cos(\log x)].$$

(Q85)

$\therefore\ [D^2 + 4D + 3]\,y = (1 + e^{-z})^2 z$

$\therefore$ The A.E. is $D^2 + 4D + 3 = 0.$

$\therefore\ (D + 1)(D + 3) = 0 \quad \therefore\ D = -1, -3.$

$\therefore$ The C.F. is $y = c_1 e^{-z} + c_2 e^{-3z}.$

$$\text{P.I.} = \frac{1}{D^2 + 4D + 3}\,(z + 2e^{-z} z + e^{-2z} z)$$

Now,
$$\frac{1}{D^2 + 4D + 3}\, z = \frac{1}{3}\left[1 + \frac{4D + D^2}{3}\right]^{-1} z$$
$$= \frac{1}{3}\left[1 - \frac{4D}{3}\ldots\ldots\right] z = \frac{1}{3}\left[z - \frac{4}{3}\ldots\ldots\right]$$

$$\frac{1}{D^2 + 4D + 3}\, 2e^{-z} z = 2e^{-z} \cdot \frac{1}{(D-1)^2 + 4(D-1) + 3}\, z$$
$$= 2 \cdot \frac{e^{-z}}{D^2 + 2D}\, z = 2 \cdot \frac{e^{-z}}{2D}\left[1 + \frac{D}{2}\ldots\ldots\right]^{-1} z$$
$$= \frac{e^{-z}}{D}\left[z - \frac{1}{2}\right] = e^{-z}\int\left[z - \frac{1}{2}\right] dz = e^{-z}\left(\frac{z^2}{2} - \frac{z}{2}\right)$$

$$\frac{1}{D^2 + 4D + 3}\, e^{-2z} \cdot z = e^{-2z} \cdot \frac{1}{(D-2)^2 + 4(D-2) + 3}\, z$$
$$= \frac{e^{-2z}}{D^2 - 1}\, z = e^{-2z} \cdot (-1)\,[1 - D^2]^{-1}\, z$$
$$= -\,e^{-2z}\,[1 + D^2 + \ldots\ldots]\, z = -\,e^{-2z} z.$$

$$\therefore\ \text{P.I.} = \frac{z}{3} - \frac{4}{9} + e^{-z}\left(\frac{z^2}{2} - \frac{z}{2}\right) - e^{-2z} \cdot z$$

$\therefore$ The complete solution is

$$y = c_1 e^{-z} + c_2 e^{-3z} + \frac{z}{3} - \frac{4}{9} + e^{-z}\left(\frac{z^2}{2} - \frac{z}{2}\right) - e^{-2z} \cdot z$$

$$y = \frac{c_1}{x} + \frac{c_2}{x^3} + \frac{\log x}{3} - \frac{4}{9} - \frac{1}{x}\left[\frac{(\log x)^2}{2} - \frac{(\log x)}{2}\right] - \frac{1}{x^2} \cdot \log x.$$

Ex. 81 : Solve $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = \log x \cdot \sin(\log x)$.

: Putting $z = \log x$ and $x = e^z$, we get

$$[D(D-1) + D + 1]y = z\sin z$$

$$\therefore \ [D^2 + 1]y = z\sin z$$

$\therefore$ The A.E. is $D^2 + 1 = 0 \quad \therefore \ D = i, -i$

$\therefore$ The C.F. is $y = c_1 \cos z + c_2 \sin z$.

$$\text{P.I.} = \frac{1}{D^2+1} z\sin z$$

$$= \text{I.P. of } \frac{1}{D^2+1} e^{iz} \cdot z$$

$$= \text{I.P. of } e^{iz} \frac{1}{(D+i)^2+1} z$$

$$= \text{I.P. of } e^{iz} \frac{1}{D^2+2iD} \cdot z$$

$$= \text{I.P. of } e^{iz} \cdot \frac{1}{2iD}\left[1+\frac{D}{2i}\right]^{-1} \cdot z$$

$$= \text{I.P. of } e^{iz} \cdot \frac{1}{2iD}\left[1-\frac{D}{2i}+\ldots\ldots\right] z$$

$$= \text{I.P. of } e^{iz} \cdot \frac{1}{2iD}\left[z-\frac{1}{2i}\right]$$

$$= \text{I.P. of } e^{iz} \cdot \frac{1}{2i}\int\left(z-\frac{1}{2i}\right) dz$$

$$= \text{I.P. of } e^{iz} \cdot \frac{1}{2i}\left[\frac{z^2}{2}-\frac{z}{2i}\right]$$

$$= \text{I.P. of } (\cos z + i\sin z)\ \frac{1}{2i}\left(\frac{z^2}{2}-\frac{z}{2i}\right)$$

$$= \text{I.P. of } (\cos z + i\sin z)\left(-\frac{i}{2}\right)\left(\frac{z^2}{2}+\frac{zi}{2}\right)$$

$$= -\frac{z^2}{4}\cos z + \frac{z}{4}\sin z$$

ie complete solution is

$$y = c_1\cos z + c_2\sin z - \frac{z^2}{4}\cos z + \frac{z}{4}\sin z$$

$$y = c_1\cos(\log x) + c_2\sin(\log x)$$

$$-\frac{(\log x)^2}{4}\cos(\log x) + \frac{(\log x)}{4}\sin(\log x).$$

07 : Solve $\left(\frac{d}{dx}+\frac{1}{x}\right)^2 y=\frac{1}{x^4}$

have

$$\left(\frac{d}{dx}+\frac{1}{x}\right)\left(\frac{dy}{dx}+\frac{y}{x}\right)=\frac{1}{x^4}$$

$$\therefore \frac{d}{dx}\left(\frac{dy}{dx}+\frac{y}{x}\right)+\frac{1}{x}\left(\frac{dy}{dx}+\frac{y}{x}\right)=\frac{1}{x^4}$$

$$\therefore \frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}-\frac{y}{x^2}+\frac{1}{x}\frac{dy}{dx}+\frac{y}{x^2}=\frac{1}{x^4}$$

$$\therefore \frac{d^2y}{dx^2}+\frac{2}{x}\frac{dy}{dx}=\frac{1}{x^4}$$

Multiplying by x^2, we get,

$$x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}=\frac{1}{x^2}$$

Putting $z=\log x$ and $x=e^z$, we get

$$[D(D-1)+2D]y=e^{-2z}$$

$\therefore$ The A.E. is $D^2+D=0$.

$\therefore D(D+1)=0 \quad \therefore D=0,-1$

$\therefore$ The C.F. is $y=c_1+c_2e^{-z}$.

$$\text{P.I.}=\frac{1}{D(D+1)}e^{-2z}=\frac{1}{-2(-2+1)}e^{-2z}=\frac{1}{2}e^{-2z}$$

$\therefore$ The complete solution is

$$y=c_1+c_2e^{-z}+\frac{1}{2}e^{-2z}$$

$$\therefore y=c_1+\frac{c_2}{x}+\frac{1}{2x^2}.$$

10 $(1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 2 \sin [\log (1+x)]$ **(Dec. 95, Dec. 97, 6**

:

Let $x + 1 = t$

Equation becomes,

$$t^2 \frac{d^2y}{dt^2} + t\frac{dy}{dt} + y = 2 \sin (\log t)$$

Let $z = \log t \quad \therefore \quad t = e^z \qquad \theta \equiv d/dz$ and $D = d/dt$

and $t^r D^r = \theta(\theta-1) \ldots (\theta - r + 1) \quad \therefore$ Equation becomes

$$[\theta(\theta-1) + \theta + 1]\, y = 2 \sin z$$

$$\therefore \quad (\theta^2 + 1)\, y = 2 \sin z$$

C.F. $\Rightarrow \quad y = c_1 \cos z + c_2 \sin z$

P.I. $\Rightarrow \quad y = \frac{1}{\theta^2 + 1}(2 \sin z)$

$$= 2\,\frac{(-1)^1}{1!}\,\frac{z^1}{(2 \cdot 1)^1} \sin (z + \pi/2) = -z \cos z$$

G.S. $\Rightarrow \quad y = c_1 \cos [\log(1+x)] + c_2 \sin [\log(1+x)]$

$$- \log(x+1) \cdot \cos [\log(1+x)]$$

Q.9: Solve $(3x+2)^2 \dfrac{d^2 y}{d x^2} + 3(3x+2)\dfrac{dy}{dx} - 36y = 3x^2 + 4x + 1$

ol. :

Let $3x + 2 = t$

$$\therefore \quad \frac{d^r y}{d x^r} = 3^r \frac{d^r y}{d t^r}$$

$$9\ t^2 \frac{d^2 y}{d t^2} + 3.3\ t \cdot \frac{dy}{dt} - 36\ y = 3\left(\frac{t-2}{3}\right)^2 + 4\left(\frac{t-2}{3}\right) + 1$$

Let $D = \dfrac{d}{dt}$

$$\therefore \quad (t^2 D^2 + t D - 4)\,y = \frac{1}{27}(t^2 - 1)$$

Let $z = \log t$,

$$t = e^z,\ \theta \equiv \frac{d}{dz},$$

$$t^r D^r \equiv \theta(\theta-1)\ .....\ (\theta - r + 1), \text{ we get,}$$

$$[\theta(\theta-1) + \theta - 4]\,y = \frac{1}{27}(e^{2z} - 1)$$

$$(\theta^2 - 4)\,y = \frac{1}{27}(e^{2z} - 1)$$

$$\text{C.F. is} \quad y = c_1\ e^{2z} + c_2\ e^{-2z}$$

$$\text{P.I. is} \quad y = \frac{1}{27} \cdot \frac{1}{\theta^2 - 4}(e^{2z} - 1) = \frac{1}{27}\left[\frac{z}{4} e^{2z} + \frac{1}{4}\right]$$

$$= \frac{1}{108}\left[z\,e^{2z} + 1\right] \quad ...($$

General Solution is from (i) and (ii),

$$y = c_1\,e^{2z} + c_2\,e^{-2z} + \frac{1}{108}\left[z\,e^{2z} + 1\right]$$

i.e. $$y = c_1(3x+2)^2 + c_2 \frac{1}{(3x+2)^2}$$

$$+ \frac{1}{108}\left[(3x+2)^2 \log(3x+2) + 1\right] \text{ is the General Solution.}$$